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# Derivative corrections to Dirac-Born-Infeld and Chern-Simon actions from Non-commutativity

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**Abstract**

We show that the higher order derivative $\alpha^{'}$ corrections to the DBI and Chern-Simon action is derived from non-commutativity in the Seiberg-Witten limit, and is shown to agree with Wyllard's (hep-th/0008125) result, as conjectured by Das et al., (hep-th/0106024). In calculating the corrections, we have expressed $\hat{F}$ in terms F, $\hat{A}$ in terms of A up to order $\mathcal{O}(A^3)$, and made use of it.

# 1 Introduction

There are several interesting developments has taken place in recent years, one of them is non-commutativity in position coordinates. In string theory, if we take a Dp brane in a flat background metric $g_{ij}$ and suspend it in a constant second rank antisymmetric tensor $B_{ij}$ field background, then one realizes a non-commutative string theory[1, 2], i.e. the ends of the open string that ends on Dp brane satisfy the following non-commutative algebra, $[X^i, X^j] = i\theta^{ij}$. Where $X^i$'s are the coordinate of the open string, and $\theta^{ij}$ is a function of the $g_{ij}, B_{ij}$[2], the background fields.

In string theory with the above mentioned background, we know that there exists two different kind of descriptions, namely, commutative and non-commutative theory. These different kind of theories arises depending on the kind of regularisation schemes that we adopt. This can be seen as: the interaction of the gauge field with the string world sheet is gauge invariant, under the gauge transformation $\delta A_i = \partial_i\lambda$, at the tree as well as the loop level in the Pauli-Villar regularisation scheme. If we shall adopt the Point-Splitting regularisation scheme instead of the Pauli-Villar regularisation scheme then the above mentioned interaction is gauge invariant only if the form of the gauge transformation is $\hat{\delta}_{\hat{\lambda}}\hat{A}_i = \partial_i\hat{\lambda} + i[\hat{\lambda}, \hat{A}_i]_*$ [2]. Moreover, it is well-known that, in quantum field theory, different regularisation schemes do not yield different S-matrix elements, also, the S-matrix element is unchanged under field redefinition in the effective action. Although, we are dealing with two seemingly different kind of descriptions but actually they are equivalent as discussed in [2], which can be realized by the well-known Seiberg-Witten map, and it implies that the actions described by the above two descriptions are related up to total derivative modulo field redefinition, i.e. $\hat{S} - S =$Total derivative $+\mathcal{O}(\partial F)$, in the DBI approximation. So, these two ways of describing the same theory can be written in a more general way[2, 3], in which the parameters of the open string, g, B, $g_s$, and closed strings, G, $\theta$, $G_s$, are related as,

$$\frac{1}{g + 2\pi\alpha' B} = \frac{1}{G + 2\pi\alpha'\Phi} + \frac{\theta}{2\pi\alpha'}$$
$$G_s = g_s\sqrt{\frac{det(G + 2\pi\alpha'\Phi)}{det(g + 2\pi\alpha' B)}} \tag{1.1}$$

Where $\Phi$ is a two form. This way of describing the theory is useful in the sense that one can describe the commutative and non-commutative theory at one stroke. Throughout our calculations we shall deal with the matrix model description, namely, $\Phi = -B$, $\theta = \frac{1}{B}$, the value of $G_s$ and G can be determined from the above equation.

To find the gauge invariant coupling of the bulk modes with the gauge fields one needs to introduce Wilson lines[6]. The Wilson line is defined as:

$$W(x,\mathcal{C}) = P_* e^{i\int_0^1 d\sigma \partial_\sigma \xi^i(\sigma)\hat{A}_i(x+\xi(\sigma))} \tag{1.2}$$

Where $\xi^i(\sigma) = \theta^{ij}k_j\sigma$, i.e. a straight Wilson line $\mathcal{C}$, and $P_*$ is the path ordering with respect to $*$ product, and $\hat{A}_i$ is the gauge potential in non-commutative space. In passing, we should mention that comparison between the R-R couplings in different descriptions yields the Seiberg-Witten map and the other topological identities[8].

We know that the low energy limits of the string theory on the brane is described by an effective theory in the $\alpha' \to 0$ limit of the string theory and the effective action has two parts, DBI and Chern-Simon actions. When $\alpha' \neq 0$ then one expects to have $\alpha'$ corrections to the action, and these corrections might be useful in the study of dualities.

In this paper, we shall verify the conjecture made by Das et al., that the derivative corrections to the Chern-Simon and the DBI action can be derived from non-commutativity.

Recently, it has been conjectured[10] that derivative corrections to DBI and Chern-Simon action can be found from non-commutativity, and the calculation has already been done to check it, to some order in F. In this report, we have not only extended this calculation but have presented the form of the Seiberg-Witten map to $\mathcal{O}(A^3)$, namely, we have expressed $\hat{F}$ in terms of F and $\hat{A}$ in terms of A to order $A^3$. We should mention en passant that it's important to know the Seiberg-Witten map, because the non-commutative action is written in non-commutative variables, $\hat{F}$ and $\hat{A}$, but to make a comparison with [9], we have to express all the terms in commutative variables i.e. in terms of F and A.

As we shall try to check this conjecture by calculating the 4-derivative corrections to the $F^3$ term for DBI action and 4-form 4-derivative corrections at $F^4$, 6-form 4-derivative corrections at $F^4$, 8-form 8-derivative corrections at $F^4$ to the Chern-Simon action. Since Das et al., has already derived the 4-derivative corrections at $F^2$ to DBI action and 4-form 4-derivative corrections at $F^3$, 6-form 6-derivative corrections at $F^3$, and 8-form 8-derivative corrections at $F^4$ to the Chern-Simon action. Evaluating the derivative corrections to Chern-Simon and DBI action, calculated by Wyllard[9], in the Seiberg-Witten limit shows an agreement of result coming from non-commutativity.

The plan of the paper is as follows: In section 2, we shall calculate the 4-derivative corrections at $F^3$ along with the 4-derivative corrections at $F^2$ to DBI action, and in section 3, we shall calculate the derivative corrections to Chern-Simon action, and conclude in section 4. We shall derive the Seiberg-Witten map and the kernels of the $*_n$ product of n functions in position space in appendix A and B respectively.

## 2 Corrections to the DBI action

The correction to the DBI action, has been calculated [9] using boundary state technique, and it is:

$$S_{DBI} = \frac{1}{g_s}\int\sqrt{det(g+2\pi\alpha'(B+F))}[1+\frac{(2\pi\alpha')^4}{96}(-h^{ij}h^{kl}h^{mn}h^{pq}S_{npjk}S_{qmli} + \frac{1}{2}h^{ij}h^{kl}S_{jk}S_{li})] \tag{2.3}$$

Where $S_{npjk} = \partial_n\partial_p F_{jk} + 2.2\pi\alpha' h^{rs}\partial_n F_{jr}\partial_p F_{ks}$ and $S_{jk} = h^{mn}S_{jkmn}$, and $h^{ij} = (\frac{1}{g+2\pi\alpha'(B+F)})^{-1ij}$, and det g is the determinant of matrix $g_{ij}$.
Note: We shall use $S_{ij}$ and a two form $\mathbf{S}_{ij}$ in this paper and they are not same, will be defined later. Writing the square-root of the determinant as,

$$\sqrt{det(g+2\pi\alpha'(B+F))} = \sqrt{det(g+2\pi\alpha' B)}\sqrt{det(1+2\pi\alpha' NF)} \tag{2.4}$$

Where N is defined in eq. (2.10). Let's evaluate the above action in the Seiberg-Witten limit, where the Seiberg-Witten limit is defined in eq.(2.8). Using $2\pi\alpha' h|_{SW} = (1+\theta F)^{-1}\theta$, and keeping terms to order $A_i^3$, we get:

$$\begin{aligned} S_{DBI}|_{SW} = &-\frac{1}{96 g_s}\int\sqrt{det(g+2\pi\alpha' B)}[\theta^{ij}\theta^{kl}\theta^{mn}\theta^{pq}\{\partial_n\partial_p F_{jk}\partial_q\partial_m F_{li} \\ &-\frac{1}{2}\partial_j\partial_k F_{mn}\partial_l\partial_i F_{pq}\} - \theta^{ij}\theta^{kl}\theta^{mn}\theta^{pq}\theta^{rs}\{2F_{sq}\partial_n\partial_p F_{jk}\partial_r\partial_m F_{li} \\ &-F_{sq}\partial_j\partial_k F_{mn}\partial_l\partial_i F_{pr} + 2F_{sl}\partial_n\partial_p F_{jk}\partial_q\partial_m F_{ri} - F_{sl}\partial_j\partial_k F_{mn}\partial_r\partial_i F_{pq} \\ &-4\partial_n\partial_p F_{jk}\partial_q F_{lr}\partial_m F_{is} + 2\partial_j\partial_k F_{mn}\partial_l F_{pr}\partial_i F_{qs} + \frac{1}{2}F_{rs}\partial_n\partial_p F_{jk}\partial_q\partial_m F_{li} \\ &-\frac{1}{4}F_{rs}\partial_j\partial_k F_{mn}\partial_l\partial_i F_{pq}\}] \end{aligned} \tag{2.5}$$

We obtain this equation by expanding eq.(2.3) and keeping terms up to 4-derivatives in $F^3$ in Seiberg-Witten limit. The sources of getting $F^3$ terms are: from product of two S's, from the $2\pi\alpha' h$ with two S's and from the determinant factor with two S's from eq.(2.3). There could be more terms in the derivative corrections to the DBI action which contributes to the 4-derivatives in $F^3$, e.g. the terms with 6 h's and 3 S's have $F^3$ terms in it, but the number of $\theta$'s that appear in the Seiberg-Witten limit are different. So, we are interested only in the situation, where we have maximum of 5 $\theta$'s and 4-derivative in $F^3$. Now, let us check that this derivative corrections can also be derived from non-commutativity, as conjectured by Das et. al. The DBI action in the commutative and non-commutative theory are[7]:

$$S_{DBI} = T_9 \int \sqrt{det(g + 2\pi\alpha'(B + F))} \tag{2.6}$$

$$\hat{S}_{DBI} = T_9 \int L_*[\frac{PfQ}{Pf\theta}\sqrt{det(g + 2\pi\alpha' Q^{-1})}W(x, \mathcal{C})] * e^{ik.x} \tag{2.7}$$

Where $W(x, \mathcal{C})$ is a straight open Wilson line with momentum k as defined in eq. (1.2). $L_*$ is defined as smearing the operators along the Wilson and taking the path ordering with respect to $*$ product. $Q^{ij} = (\theta - \theta\hat{F}\theta)^{ij}$, $Q^{-1} = \theta^{-1} + \hat{F}(1 - \theta\hat{F})^{-1}$, and $Pf\theta = \sqrt{det\theta}$.

Here, we are dealing with a space-filling brane, to avoid the appearance of scalars through pull-back. The Seiberg-Witten limit is defined as,

$$\alpha' \sim \sqrt{\epsilon} \to 0, \quad g_{ij} \sim \epsilon \to 0, \quad \text{holding} \;\; G_{ij},\; B_{ij},\; G_s \;\; \text{fixed}. \tag{2.8}$$

The prescription to find the derivative corrections to the DBI (and Chern-Simon action) is to evaluate the difference between the DBI action in non-commutative and commutative theory in the Seiberg-Witten limit, $\hat{S}_{DBI}|_{SW} - S_{DBI}|_{SW}$. Note, eq.(2.7) is in momentum space, but we shall do all the calculations in position space throughout the paper. The tension of the space filling brane is $T_9 \sim \frac{1}{g_s}$. Hence Eq.(2.6) can be written as

$$\frac{1}{g_s}\int \sqrt{det(g + 2\pi\alpha' B)}\sqrt{det(1 + 2\pi\alpha' NF)} \tag{2.9}$$

where N is defined as:

$$N^{ij} \equiv (\frac{1}{g + 2\pi\alpha' B})^{ij} = \frac{\theta^{ij}}{2\pi\alpha'} + (\frac{1}{G + 2\pi\alpha'\Phi})^{ij} \tag{2.10}$$

In the Seiberg-Witten limit, $N^{ij}|_{SW} \to \frac{\theta^{ij}}{2\pi\alpha'}$. Hence, the commutative DBI action eq.(2.6) in this limit becomes,

$$\frac{1}{g_s}\int \sqrt{det(g + 2\pi\alpha' B)}\sqrt{det(1 + \theta F)} \tag{2.11}$$

Where as the corresponding non-commutative DBI action eq.(2.7) can be rewritten as:

$$\frac{1}{g_s}\int L_*[\sqrt{det(1 - \theta\hat{F})}\sqrt{det(g + 2\pi\alpha' B)}\sqrt{det(1 + 2\pi\alpha' N\hat{F}\frac{1}{1 - \theta\hat{F}})}W(x, \mathcal{C})] * e^{ik.x} \tag{2.12}$$

In the Seiberg-Witten limit this non-commutative DBI action becomes,

$$\frac{1}{g_s}\int L_*[\sqrt{det(g + 2\pi\alpha' B)}W(x, \mathcal{C})] * e^{ik.x} \tag{2.13}$$

Therefore, in order to find the derivative corrections we have to find the difference between eq.(2.13) and eq.(2.11). But its not easy to find the difference, since one of the equation is written using non-commutative variables where as the other one is in commutative variables. In order to find the difference we shall use the Seiberg-Witten map, to convert the non-commutative variables into its commutative form.

Eq.(2.11) can be written up to $\mathcal{O}(A^3)$, as

$$\begin{aligned} S_{DBI}|_{SW} &= \frac{1}{g_s}\int\sqrt{det(g+2\pi\alpha' B)}[1+\frac{1}{2}tr(\theta F)-\frac{1}{4}tr(\theta F)^2+\frac{1}{8}(tr(\theta F))^2 \\ &\quad +\frac{1}{6}tr(\theta F)^3-\frac{1}{8}tr(\theta F)tr(\theta F)^2+\frac{1}{48}(tr(\theta F))^3] \end{aligned} \tag{2.14}$$

Rearranging the terms, we get:

$$\begin{aligned} S_{DBI}|_{SW} &= \frac{1}{g_s}\int\sqrt{det(g+2\pi\alpha' B)}[1-\theta^{ij}\partial_i A_j-\frac{\theta^{ij}\theta^{kl}}{2}(\partial_j A_k\partial_l A_i-\partial_j A_k\partial_i A_l \\ &-\partial_i A_j\partial_k A_l)+\frac{\theta^{ij}\theta^{kl}\theta^{mn}}{6}(2\partial_j A_k\partial_l A_m\partial_n A_i-2\partial_j A_k\partial_l A_m\partial_i A_n-2\partial_k A_j\partial_l A_m\partial_n A_i \\ &+\frac{6}{4}\{2\partial_i A_j\partial_l A_m\partial_n A_k-2\partial_i A_j\partial_l A_m\partial_k A_n\}-\partial_i A_j\partial_k A_l\partial_m A_n)] \end{aligned} \tag{2.15}$$

On expanding the equation(2.13) to the order we are interested in, to $\mathcal{O}(A^3)$, we get:

$$\begin{aligned} \hat{S}_{DBI}|_{SW} &= \frac{1}{g_s}\int\sqrt{det(g+2\pi\alpha' B)}[1+\theta^{ij}\partial_j\hat{A}_i+\frac{1}{2}\theta^{ij}\theta^{kl}\partial_j\partial_l\langle\hat{A}_i,\hat{A}_k\rangle \\ &\quad +\frac{1}{6}\theta^{ij}\theta^{kl}\theta^{mn}\partial_j\partial_l\partial_n\langle\hat{A}_i,\hat{A}_k,\hat{A}_m\rangle] \end{aligned} \tag{2.16}$$

Substituting the Seiberg-Witten map of the potential[1] into the above equation gives us a large number of terms, we shall divide them according to the power of $A_i$. Then, we shall find the difference between the commutative and non-commutative action to different order in $A_i$.

<u>$\hat{S}_{DBI}|_{SW}-S_{DBI}|_{SW}$ to order $A_i^2$.</u>

The non-commutative, and commutative DBI action to the order we are working in is:

$$\begin{aligned} &\hat{S}_{DBI}|_{SW}=\frac{1}{g_s}\int\sqrt{det(g+2\pi\alpha' B)}[1+\theta^{ij}\partial_j A_i-\frac{\theta^{ij}\theta^{kl}}{2}\{\langle\partial_j A_k,\partial_l A_i\rangle-\langle\partial_j A_k,\partial_i A_l\rangle \\ &-\langle\partial_j A_i,\partial_l A_k\rangle\}] \\ &S_{DBI}|_{SW}=\frac{1}{g_s}\int\sqrt{det(g+2\pi\alpha' B)}[1-\theta^{ij}\partial_i A_j-\frac{\theta^{ij}\theta^{kl}}{2}(\partial_j A_k\partial_l A_i-\partial_j A_k\partial_i A_l \\ &-\partial_i A_j\partial_k A_l)] \end{aligned} \tag{2.17}$$

[1]The Seiberg-Witten map is given in Appendix A

and the difference between them is :

$$\hat{S}_{DBI}|_{SW} - S_{DBI}|_{SW} = \frac{1}{g_s}\int\sqrt{det(g+2\pi\alpha' B)}[-\frac{\theta^{ij}\theta^{kl}}{2}\{\frac{1}{2}\langle F_{jk},F_{li}\rangle - \frac{1}{2}F_{jk}F_{li} - \frac{1}{4}\langle F_{ij},F_{kl}\rangle + \frac{1}{4}F_{ij}F_{kl}\} \tag{2.18}$$

Substituting the expression of $\langle f,g\rangle$ in the above equation, we get to 8-derivative in the field strength, F, as:

$$\hat{S}_{DBI}|_{SW} - S_{DBI}|_{SW} = \frac{1}{g_s}\int\sqrt{det(g+2\pi\alpha' B)}[-\frac{\theta^{ij}\theta^{kl}}{2}\{\frac{\theta^{mn}\theta^{pq}}{48}(\partial_m\partial_p F_{jk}\partial_n\partial_q F_{li} - \frac{1}{2}\partial_m\partial_p F_{ij}\partial_n\partial_q F_{kl}) + \frac{\theta^{mn}\theta^{pq}\theta^{rs}\theta^{uv}}{3840}(\partial_m\partial_p\partial_r\partial_u F_{jk}\partial_n\partial_q\partial_s\partial_v F_{li} - \frac{1}{2}\partial_m\partial_p\partial_r\partial_u F_{ij}\partial_n\partial_q\partial_s\partial_v F_{kl})\}] \tag{2.19}$$

The difference between these two actions at the next order, at $A_i^3$ is:

$$\hat{S}_{DBI}|_{SW} - S_{DBI}|_{SW} = \frac{1}{g_s}\int\sqrt{det(g+2\pi\alpha' B)}[\frac{\theta^{ij}\theta^{kl}\theta^{mn}\theta^{pq}\theta^{rs}}{24} \{\frac{1}{2}\partial_p\partial_r F_{jk}\partial_q\partial_s F_{lm}F_{ni} + \frac{1}{4}\partial_p\partial_r F_{ij}\partial_q\partial_s F_{lm}F_{nk} + \frac{1}{8}F_{ij}\partial_p\partial_r F_{lm}\partial_q\partial_s F_{nk} - \frac{1}{16}\partial_p\partial_r F_{ij}\partial_q\partial_s F_{kl}F_{mn} - \frac{1}{4}F_{pm}\partial_r\partial_n F_{lk}\partial_q\partial_s F_{ji} + \frac{1}{2}\partial_p F_{lm}\partial_r F_{nk}\partial_q\partial_s F_{ij} + \frac{1}{2}F_{pk}\partial_q\partial_s F_{jm}\partial_l\partial_r F_{ni} - \partial_p F_{kj}\partial_r F_{ml}\partial_q\partial_s F_{in}\}] \tag{2.20}$$

We can see that the equation (2.5) is same as the corrections that we found from non-commutativity, namely, the sum of equation (2.19) and equation (2.20), up to 4-derivative in $F^3$.

# 3 Corrections to the Chern-Simon action

The Chern-Simon action in two different descriptions, namely in the $\theta = 0, \Phi = B$ and $\Phi = -B, \theta = \frac{1}{B}$, are as follows[7]:

$$S_{CS} = \frac{1}{g_s}\int\sum_n C^{(n)} \wedge e^{2\pi\alpha'(B+F)} \tag{3.21}$$

$$\hat{S}_{CS} = \frac{1}{g_s}\int e^{ik.x} * L_*[\frac{PfQ}{Pf\theta}\sum_n C^{(n)} \wedge e^{2\pi\alpha' Q^{-1}}W(x,\mathcal{C})] \tag{3.22}$$

The Non-commutative Chern-Simon action is written in momentum space, but while calculating the difference between the above two action, we shall do so in position space. To make comparison with the results of [9], we shall parametrise the Chern-Simon action with correction as follows:

$$S_{CS} = \frac{1}{g_s} \int \sum_n C^{(n)} \wedge e^{2\pi\alpha'(B+F)} \wedge e^{\sum_{k=2}^{5} \mathcal{W}_{2k}} \tag{3.23}$$

Where $\mathcal{W}_{2k}$ is a 2k form and function of derivatives of field strength and $2\pi\alpha'(B+F)$. As has been pointed out in [10], the $\mathcal{W}$'s that we shall use are not exactly same as the W's of [9], e.g. the $W_8$ of [9] is $\mathcal{W}_8 + \frac{1}{2}\mathcal{W}_4 \wedge \mathcal{W}_4$ of ours. For completeness, we shall mention the $\mathcal{W}$'s that we shall use to calculate the corrections, and they are:

$$\begin{aligned}
\mathcal{W}_4 \equiv \frac{W_4}{(2\pi\alpha')^2} &= (2\pi\alpha')^2 \frac{\zeta(2)}{8\pi^2} h^{ij} h^{kl} \mathbf{S}_{jk} \wedge \mathbf{S}_{li} \\
\mathcal{W}_6 \equiv \frac{W_6}{(2\pi\alpha')^3} &= (2\pi\alpha')^3 \frac{\zeta(3)}{24\pi^3} h^{ij} h^{kl} h^{mn} \mathbf{S}_{lm} \wedge \mathbf{S}_{jk} \wedge \mathbf{S}_{ni} \\
\mathcal{W}_8 \equiv \frac{W_8}{(2\pi\alpha')^4} &= (2\pi\alpha')^4 \frac{\zeta(4)}{64\pi^4} h^{ij} h^{kl} h^{mn} h^{pq} \mathbf{S}_{np} \wedge \mathbf{S}_{lm} \wedge \mathbf{S}_{jk} \wedge \mathbf{S}_{qi}
\end{aligned} \tag{3.24}$$

Where the $\mathbf{S}_{ij}$ is a two form and defined as

$$\begin{aligned}
\mathbf{S}_{ij} &= \frac{1}{2} S_{ijab} dx^a \wedge dx^b, \quad with \\
S_{ijab} &= \partial_i \partial_j F_{ab} + 2.2\pi\alpha' h^{kl} \partial_i F_{ak} \partial_j F_{bl}
\end{aligned} \tag{3.25}$$

and $\zeta(n)$ are the Riemann-Zeta functions. The matching of results will be done only in the Seiberg-Witten limit.

In our study of derivative corrections, we shall deal with IIB string theory, mainly with a D9 brane, and consider the interaction of this brane with even R-R form potentials.

Interaction with $C^{(10)}$:

The difference between the two actions (3.21) and (3.22) for this case gives rise to a topological identity, which is[8]:

$$\delta(k) = \int dx L_* [\sqrt{det(1 - \theta\hat{F})} W(x, \mathcal{C})] * e^{ik.x} \tag{3.26}$$

Interaction with $C^{(8)}$:

On finding the difference here, between (3.21) and (3.22) gives us the Seiberg-Witten map[4, 8].

$$F(k) = \int dx L_* [\sqrt{det(1 - \theta\hat{F})} \hat{F} (1 - \theta\hat{F})^{-1} W(x, \mathcal{C})] * e^{ik.x} \tag{3.27}$$

Interaction with $C^{(6)}$:

We shall do the comparison with [9] to the derivative corrections in the Seiberg-Witten limit. As we know a D9-brane can interact with a $C^{(6)}$ R-R potential through $(B+F)\wedge(B+F)$, the commutative and non-commutative Chern-Simon actions, in this case, are:

$$\begin{aligned}\hat{S}_{CS} &= \frac{1}{2g_s}\int L_*[\frac{PfQ}{Pf\theta}C^{(6)}\wedge 2\pi\alpha' Q^{-1}\wedge 2\pi\alpha' Q^{-1}W(x,\mathcal{C})]*e^{ik.x} \\ S_{CS} &= \frac{1}{2g_s}\int C^{(6)}\wedge 2\pi\alpha'(B+F)\wedge 2\pi\alpha'(B+F) \end{aligned} \tag{3.28}$$

The difference between these two actions, in position space, to order $\mathcal{O}(A^4)$ is:

$$\begin{aligned}&\Delta\mathcal{S}_{CS} \equiv \frac{\Delta S_{CS}}{(2\pi\alpha')^2} = \frac{1}{8g_s}\int C^{(6)}[\langle\hat{F}_{ab},\hat{F}_{cd}\rangle + 2\theta^{ij}\langle\hat{F}_{ab},\hat{F}_{ci},\hat{F}_{jd}\rangle + \frac{1}{2}\theta^{ij}\langle\hat{F}_{ij},\hat{F}_{ab},\hat{F}_{cd}\rangle + \\ &\theta^{ij}\partial_j\langle\hat{F}_{ab},\hat{F}_{cd},\hat{A}_i\rangle + \theta^{ij}\theta^{kl}(2\langle\hat{F}_{ab},\hat{F}_{ci},\hat{F}_{jk},\hat{F}_{ld}\rangle + \langle\hat{F}_{ai},\hat{F}_{jb},\hat{F}_{ck},\hat{F}_{ld}\rangle + \langle\hat{F}_{ij},\hat{F}_{ab},\hat{F}_{ck},\hat{F}_{ld}\rangle + \\ &\frac{1}{8}\langle\hat{F}_{ij},\hat{F}_{kl},\hat{F}_{ab},\hat{F}_{cd}\rangle - \frac{1}{4}\langle\hat{F}_{jk},\hat{F}_{li},\hat{F}_{ab},\hat{F}_{cd}\rangle + \frac{1}{2}\partial_j\partial_l\langle\hat{F}_{ab},\hat{F}_{cd},\hat{A}_i,\hat{A}_k\rangle + 2\partial_j\langle\hat{F}_{ab},\hat{F}_{ck},\hat{F}_{ld},\hat{A}_i\rangle + \\ &\frac{1}{2}\partial_j\langle\hat{F}_{kl},\hat{F}_{ab},\hat{F}_{cd},\hat{A}_i\rangle) - F_{ab}F_{cd}]dx^a\wedge dx^b\wedge dx^c\wedge dx^d \end{aligned} \tag{3.29}$$

On substituting the Seiberg-Witten map[2], $\hat{F}_{ij}$ in terms of $F_{ij}$ and $\hat{A}_i$ in terms of $A_i$ in the above equation, we get to $\mathcal{O}(A^3)$ as:

$$\begin{aligned}\Delta\mathcal{S}_{CS} = &\frac{1}{8g_s}\int C^{(6)}\wedge[\langle F_{ab},F_{cd}\rangle - F_{ab}F_{cd} \\ &-2\theta^{mn}\{\langle F_{ab},\langle A_m,\partial_n F_{cd}\rangle\rangle + \langle F_{ab},\langle F_{cm},F_{nd}\rangle\rangle - \langle F_{ab},F_{cm},F_{nd}\rangle - \langle F_{ab},\partial_n F_{cd},A_m\rangle\}] \\ &dx^a\wedge dx^b\wedge dx^c\wedge dx^d\end{aligned} \tag{3.30}$$

Where (in general) the expression $\langle f,\langle g,h\rangle_{*2}\rangle_{*2}$ is written as $\langle f,\langle g,h\rangle\rangle$ and $\langle f,g,h\rangle_{*3}$ as $\langle f,g,h\rangle$ to avoid clumsiness[3]. On substituting the expression for $*_2$, $*_2$ within $*_2$ and for $*_3$ into the above equation, we get:

$$\begin{aligned}&\Delta\mathcal{S}_{CS} = \frac{1}{8g_s}\int C^{(6)}\wedge[-\frac{\theta^{ij}\theta^{kl}}{24}\partial_i\partial_k F_{ab}\partial_j\partial_l F_{cd} - \frac{\theta^{ij}\theta^{kl}\theta^{mn}}{6} \\ &\{\frac{1}{2}F_{im}\partial_k\partial_n F_{cd}\partial_j\partial_l F_{ab} + \partial_i F_{cm}\partial_k F_{nd}\partial_j\partial_l F_{ab}\}]dx^a\wedge dx^b\wedge dx^c\wedge dx^d\end{aligned} \tag{3.31}$$

Let's compare this with the result that will come from $\mathcal{W}_4$ in the Seiberg-Witten limit, i.e. the 4-form 4-derivative correction to the Chern-Simon action in the Seiberg-Witten limit is:

$$\begin{aligned}&\mathcal{W}_4|_{SW} = \frac{1}{192}[-\theta^{ij}\theta^{kl}\partial_i\partial_k F_{ab}\partial_j\partial_l F_{cd} + 2\theta^{ij}\theta^{kl}\theta^{mn} \\ &\{F_{im}\partial_n\partial_k F_{cd}\partial_j\partial_l F_{ab} + 2\partial_i F_{cm}\partial_k F_{nd}\partial_j\partial_l F_{ab}\}]dx^a\wedge dx^b\wedge dx^c\wedge dx^d\end{aligned} \tag{3.32}$$

[2]The Seiberg-Witten map is derived in Appendix A.

[3]The exact form of $*_2$, $*_3$, *and* $*_2$ within $*_2$, is derived in Appendix B.

On inclusion of tension as well as the integration, we can very easily see that both the above equations are same.

Interaction with $C^{(4)}$:

The non-commutative and commutative Chern-Simon actions for a D9-brane that couples to a $C^{(4)}$ R-R potential are:

$$\begin{aligned}\hat{S}_{CS} &= \frac{1}{6g_s}\int L_*[\frac{PfQ}{Pf\theta}C^{(4)}\wedge 2\pi\alpha' Q^{-1}\wedge 2\pi\alpha' Q^{-1}\wedge 2\pi\alpha' Q^{-1}W(x,\mathcal{C})]*e^{ik.x} \\ S_{CS} &= \frac{1}{6g_s}\int C^{(4)}\wedge 2\pi\alpha'(B+F)\wedge 2\pi\alpha'(B+F)\wedge 2\pi\alpha'(B+F) \end{aligned} \tag{3.33}$$

In position space, the difference between the above two actions, to order $\mathcal{O}(A^4)$, is :

$$\begin{aligned}\Delta\mathcal{S}_{CS} &\equiv \frac{\Delta S_{CS}}{(2\pi\alpha')^3} = \frac{1}{48g_s}\int C^{(4)}\wedge\{\langle\hat{F}_{ab},\hat{F}_{cd},\hat{F}_{ef}\rangle - F_{ab}F_{cd}F_{ef} - \frac{1}{2}\theta^{gh}\langle\hat{F}_{gh},\hat{F}_{ab},\hat{F}_{cd},\hat{F}_{ef}\rangle \\ &+3\theta^{gh}\langle\hat{F}_{ab},\hat{F}_{cd},\hat{F}_{eg},\hat{F}_{hf}\rangle + \theta^{gh}\partial_h\langle\hat{F}_{ab},\hat{F}_{cd},\hat{F}_{ef},\hat{A}_g\rangle\}dx^a\wedge dx^b\wedge dx^c\wedge dx^d\wedge dx^e\wedge dx^f\end{aligned} \tag{3.34}$$

Using the Seiberg-Witten map in the above equation and expressing all the terms in terms of commutative variables, we get:

$$\begin{aligned}\Delta\mathcal{S}_{CS} &= \frac{1}{48g_s}\int C^{(4)}\wedge\{\langle F_{ab},F_{cd},F_{ef}\rangle - F_{ab}F_{cd}F_{ef} - 3\theta^{gh}\langle\langle A_g,\partial_h F_{ab},\rangle F_{cd},F_{ef}\rangle \\ &+3\theta^{gh}\langle\langle F_{ag},F_{bh}\rangle,F_{cd},F_{ef}\rangle + \frac{1}{2}\theta^{gh}\langle F_{gh},F_{ab},F_{cd},F_{ef}\rangle + 3\theta^{gh}\langle F_{ab},F_{cd},F_{eg},F_{hf}\rangle \\ &+\theta^{gh}\partial_h\langle F_{ab},F_{cd},F_{ef},A_g\rangle\}dx^a\wedge dx^b\wedge dx^c\wedge dx^d\wedge dx^e\wedge dx^f\end{aligned} \tag{3.35}$$

Using eq.(3.24), the corrections to the Chern-Simon action from (3.23) for this form of R-R potential is:

$$F\wedge\mathcal{W}_4 + \mathcal{W}_6 \tag{3.36}$$

Since we have to match both the results in the Seiberg-Witten limit, implies we have to evaluate the above expression in the said limit, and it becomes:

$$\begin{aligned}\mathcal{W}_6|_{SW} &= -F\wedge\mathcal{W}_4|_{SW} + \frac{1}{48}\{\langle F_{ab},F_{cd},F_{ef}\rangle - F_{ab}F_{cd}F_{ef} - 3\theta^{gh}\langle\langle A_g,\partial_h F_{ab},\rangle F_{cd},F_{ef}\rangle \\ &+3\theta^{gh}\langle\langle F_{ag},F_{bh}\rangle,F_{cd},F_{ef}\rangle + \frac{1}{2}\theta^{gh}\langle F_{gh},F_{ab},F_{cd},F_{ef}\rangle + 3\theta^{gh}\langle F_{ab},F_{cd},F_{eg},F_{hf}\rangle \\ &+\theta^{gh}\partial_h\langle F_{ab},F_{cd},F_{ef},A_g\rangle\}dx^a\wedge dx^b\wedge dx^c\wedge dx^d\wedge dx^e\wedge dx^f\end{aligned} \tag{3.37}$$

It's very straightforward to see that $\mathcal{W}_6|_{SW}$ from eq.(3.24) in the Seiberg-Witten limit vanishes due to the symmetry property. Which implies that the above equation should vanish, and on substituting the expression for $*_2$ within $*_3$ and $*_4$ in the above equation[4], we can easily confirm ourself that it vanishes in the said limit.

[4]These expressions are written in Appendix B.

Interaction with $C^{(2)}$:

The actions are,

$$
\begin{aligned}
\hat{S}_{CS} &= \frac{1}{4!g_s}\int L_*[\frac{PfQ}{Pf\theta}C^{(2)}\wedge 2\pi\alpha^{'}Q^{-1}\wedge 2\pi\alpha^{'}Q^{-1}\wedge 2\pi\alpha^{'}Q^{-1}\wedge 2\pi\alpha^{'}Q^{-1}W(x,\mathcal{C})] * e^{ik.x}\\
S_{CS} &= \frac{1}{4!g_s}\int C^{(2)}\wedge 2\pi\alpha^{'}(B+F)\wedge 2\pi\alpha^{'}(B+F)\wedge 2\pi\alpha^{'}(B+F)\wedge 2\pi\alpha^{'}(B+F)
\end{aligned}
\tag{3.38}
$$

Using the Seiberg-Witten map, the difference between them, to order $F^4$, becomes:

$$
\Delta\mathcal{S}_{CS} \equiv \frac{\Delta S_{CS}}{(2\pi\alpha')^4} = \frac{1}{4!g_s}\int C^{(2)}\wedge\{\langle F\wedge F\wedge F\wedge F\rangle - F\wedge F\wedge F\wedge F\} \tag{3.39}
$$

Using eq.(3.23) the corrections to the Chern-Simon action is parametrized as:

$$
\frac{1}{g_s}\int C^{(2)}\wedge\{\mathcal{W}_8 + F\wedge\mathcal{W}_6 + \frac{1}{2}\mathcal{W}_4\wedge\mathcal{W}_4 + \frac{1}{2}F\wedge F\wedge\mathcal{W}_4\} \tag{3.40}
$$

According to the conjecture we should match both results in the Seiberg-Witten limit, and it becomes:

$$
\begin{aligned}
\mathcal{W}_8|_{SW} &= \frac{1}{4!}\langle F\wedge F\wedge F\wedge F\rangle - \frac{1}{4!}F\wedge F\wedge F\wedge F - F\wedge\mathcal{W}_4|_{SW}\\
&\quad -\frac{1}{2}F\wedge F\wedge\mathcal{W}_4|_{SW} - \frac{1}{2}\mathcal{W}_4\wedge\mathcal{W}_4|_{SW}
\end{aligned}
\tag{3.41}
$$

Substituting the expression of $\mathcal{W}_4$ and $\mathcal{W}_6$ in the Seiberg-Witten limit into the above equation, results in, to order $\mathcal{O}(A^4)$ :

$$
\frac{\theta^{ij}\theta^{kl}\theta^{mn}\theta^{pq}}{5760}\partial_i\partial_k F_{ab}\partial_m\partial_p F_{cd}\partial_j\partial_n F_{ef}\partial_l\partial_q F_{gh}dx^a\wedge\ldots\wedge dx^h \tag{3.42}
$$

It is easy to check that the $\mathcal{W}_8$ of eq.(3.24) in the Seiberg-Witten limit reproduces the above result with the same coefficient.

# 4 Conclusion

We have demonstrated the conjecture that the derivative corrections to the commutative theory can be found from non-commutativity in the Seiberg-Witten limit. It's important to take this limit because in this limit all the corrections to the non-commutative action vanishes and left with the derivative corrections to the commutative theory. Also, to do the calculation at higher order in field strength we need to know the Seiberg-Witten map, i.e. the expression of $\hat{F}_{ij}$ in terms of $F_{ij}$ and $\hat{A}_i$ in terms of $A_i$.

Moreover, we shall explain the 4-form 4-derivative corrections to the Chern-Simon action at $F^4$. Let's explain it in detail. The corrections to the Chern-Simon action at this order, from eq.(3.24) is :

$$\begin{aligned}&\frac{\theta^{ij}\theta^{kl}\theta^{mn}\theta^{pq}}{192}[2F_{lq}F_{pn}\partial_j\partial_kF_{ab}\partial_m\partial_iF_{cd}+F_{jn}F_{lq}\partial_m\partial_kF_{ab}\partial_p\partial_iF_{cd}\\&+4F_{jq}\partial_n\partial_kF_{cd}\partial_lF_{ai}\partial_mF_{bp}+4\partial_nF_{ai}\partial_pF_{bj}\partial_qF_{ck}\partial_mF_{dl}\\&+8F_{np}\partial_j\partial_mF_{cd}\partial_iF_{bl}\partial_qF_{ak}]dx^a\wedge\ldots\wedge dx^d\end{aligned}\tag{4.43}$$

The corresponding term from non-commutativity is eq.(3.29). Using the Seiberg-Witten map, we can rewrite that term at the order we are working, as follows:

$$\begin{aligned}&\Delta\mathcal{S}_{CS}=\frac{1}{8g_s}\int C^{(6)}\wedge\theta^{ef}\theta^{gh}[-\langle\langle\partial_g\partial_eF_{cd},A_h,A_f\rangle,F_{ab}\rangle\\&-2\langle\langle\partial_gF_{cd},\partial_eA_h.A_f\rangle,F_{ab}\rangle-\langle\langle F_{cd},\partial_eA_h,p_gA_f\rangle,F_{ab}\rangle-4\langle\langle F_{cg},\partial_eF_{dh},A_f\rangle,F_{ab}\rangle\\&-\frac{1}{2}\langle\langle F_{cd},F_{he},F_{gf}\rangle,F_{ab}\rangle+2\langle\langle F_{ge},F_{cf},F_{dh}\rangle F_{ab}\rangle+2\langle F_{ab},\langle\langle A_e,\partial_fA_g\rangle,\partial_hF_{cd}\rangle\rangle\\&+\langle F_{ab},\langle\langle\partial_hA_e,\partial_gA_f\rangle,F_{cd}\rangle\rangle-\langle F_{ab},\langle\langle A_e,\partial_gA_f\rangle F_{cd}\rangle\rangle+2\langle F_{ab},\langle\langle\partial_hA_e,\partial_fF_{cd}\rangle,A_g\rangle\rangle\\&+2\langle F_{ab},\langle\langle A_e,\partial_h\partial_fF_{cd}\rangle,A_g\rangle\rangle-4\langle F_{ab},\langle\langle\partial_hF_{ce},F_{df}\rangle A_g\rangle\rangle-4\langle F_{ab},\langle\langle A_e,\partial_fF_{dh}\rangle,F_{cg}\rangle\rangle\\&+4\langle F_{ab},\langle\langle F_{de},F_{hf}\rangle,F_{cg}\rangle\rangle+\langle\langle A_e,\partial_fF_{ab}\rangle,\langle A_g,\partial_hF_{cd}\rangle\rangle-2\langle\langle A_e,\partial_fF_{ab}\rangle,\langle F_{cg},F_{dh}\rangle\rangle\\&+\langle\langle F_{ae},F_{bf}\rangle,\langle F_{cg},F_{dh}\rangle\rangle-4\langle\langle A_g,\partial_hF_{ce}\rangle,F_{ab},F_{fd}\rangle+4\langle\langle F_{cg},F_{eh}\rangle,F_{ab},F_{cd}\rangle\\&-2\langle\langle A_e,\partial_fF_{ab}\rangle,F_{cg},F_{hd}\rangle+2\langle\langle F_{ae},F_{bf}\rangle,F_{cg},F_{hd}\rangle+\frac{1}{2}\langle\langle F_{ge},F_{hf}\rangle,F_{ab},F_{cd}\rangle\\&-2\langle\langle A_g,\partial_hA_e\rangle,\partial_fF_{ab},F_{cd}\rangle-\langle\langle\partial_fA_g,\partial_hA_e\rangle,F_{ab},F_{cd}\rangle+\langle\langle A_g,\partial_eA_h\rangle,\partial_fF_{ab},F_{cd}\rangle\\&+\frac{1}{2}\langle\langle\partial_fA_g,\partial_eA_h\rangle,F_{ab},F_{cd}\rangle-2\langle\langle\partial_hA_e,\partial_fF_{ab}\rangle,F_{cd},A_g\rangle-2\langle\langle A_e,\partial_h\partial_fF_{ab}\rangle,F_{cd},A_g\rangle\\&-2\langle\langle A_e,\partial_fF_{ab}\rangle,\partial_hF_{cd},A_g\rangle+4\langle\langle\partial_hF_{ae},F_{bf}\rangle,F_{cd},A_g\rangle+2\langle\langle F_{ae},F_{bf}\rangle,\partial_hF_{cd},A_g\rangle\\&+2\langle F_{ab},F_{ce},F_{fg},F_{hd}\rangle+\langle F_{ae},F_{fb},F_{cg},F_{hd}\rangle-\frac{1}{4}\langle F_{fg},F_{he},F_{ab},F_{cd}\rangle\\&+\langle\partial_f\partial_hF_{ab},F_{cd},A_g,A_e\rangle+\langle\partial_fF_{ab},\partial_hF_{cd},A_g,A_e\rangle+2\langle F_{ab},\partial_fF_{cd},A_g,\partial_hA_e\rangle\\&+\frac{1}{2}\langle F_{ab},F_{cd},\partial_fA_g,\partial_hA_e\rangle+2\langle\partial_fF_{ab},F_{cg},F_{hd},A_e\rangle+4\langle F_{ab},\partial_fF_{cg},F_{hd},A_e\rangle]\\&dx^a\wedge\ldots\wedge dx^d\end{aligned}\tag{4.44}$$

Substituting the expression of $*_n$ from Appendix B, we see as a first check that to quadratic in $\theta$ the correction vanishes, which is in consistent with the result of [9]. The term at 4-derivative to $F^4$ is:

$$\begin{aligned}&-\frac{\theta^{ij}\theta^{kl}\theta^{mn}\theta^{pq}}{192}[4\partial_iF_{am}\partial_kF_{bn}\partial_jF_{cp}\partial_lF_{dq}-4F_{pm}\partial_iF_{cn}\partial_kF_{dq}\partial_j\partial_lF_{ab}\\&+8\partial_lF_{dq}\partial_jF_{cp}F_{mi}\partial_k\partial_nF_{ab}-F_{qi}F_{kn}\partial_j\partial_lF_{ab}\partial_m\partial_pF_{cd}+2\partial_i\partial_pF_{cd}F_{mq}F_{kn}\partial_j\partial_lF_{ab}\\&-2\partial_i\partial_kF_{ab}A_m\partial_j\partial_pA_n\partial_l\partial_qF_{cd}]dx^a\wedge\ldots\wedge dx^d\end{aligned}\tag{4.45}$$

We can see that to order $\theta^4$, the result of the calculation from non-commutativity matches with that of eq.(4.43), i.e. we reproduced all the terms that appear in eq.(4.43), but along with these terms, we have an extra term in eq.(4.45), from non-commutativity, and this extra term vanishes due to symmetry arguments.

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# 5 Appendix A

In this section we shall derive the Seiberg-Witten map, namely, expressing $\hat{F}_{ij}$ in terms of $F_{ij}$ and $\hat{A}_i$ in terms of $A_i$, by solving the equation(3.27) along with the expression of $A_i$ in terms of $\hat{A}_i$[5]. Moreover, it's easy to check that the expression of $\hat{F}_{ij}$ in terms of $F_{ij}$ is consistent with the form of $\hat{A}_i$ in terms of $A_i$. Let's expand the eq.(3.27) to order $A^3$, and writing in position space, we get the field strength as:

$$\begin{aligned}&F_{ab} = \hat{F}_{ab} + \theta^{ij}\{\partial_j\langle\hat{A}_i, \hat{F}_{ab}\rangle + \frac{1}{2}\langle\hat{F}_{ab}, \hat{F}_{ij}\rangle - \langle\hat{F}_{ai}, \hat{F}_{bj}\rangle\}\\ &+\frac{1}{2}\theta^{ij}\theta^{kl}\{\partial_i\partial_k\langle\hat{F}_{ab}, \hat{A}_l, \hat{A}_j\rangle - \partial_k\langle\hat{F}_{ij}, \hat{F}_{ab}, \hat{A}_l\rangle + 2\partial_k\langle\hat{F}_{ai}, \hat{F}_{bj}, \hat{A}_l\rangle\}\\ &-\theta^{ij}\theta^{kl}\{\frac{1}{2}\langle\hat{F}_{ai}, \hat{F}_{bj}, \hat{F}_{kl}\rangle - \frac{1}{8}\langle\hat{F}_{ab}, \hat{F}_{kl}, \hat{F}_{ij}\rangle - \frac{1}{4}\langle\hat{F}_{ab}, \hat{F}_{jk}, \hat{F}_{il}\rangle + \langle\hat{F}_{ik}, \hat{F}_{al}, \hat{F}_{bj}\rangle\} + \mathcal{O}(A^4)\end{aligned} \tag{5.46}$$

The corresponding Seiberg-Witten map of the potential, to order $\mathcal{O}(A^3)$ is:

$$\begin{aligned}A_b \;=\; &\hat{A}_b + \frac{1}{2}\theta^{ij}\langle\hat{A}_i, (\partial_j\hat{A}_b + \hat{F}_{jb})\rangle + \frac{1}{2}\theta^{ij}\theta^{kl}[-\langle\hat{A}_i, \partial_k\hat{A}_b, (\partial_j\hat{A}_l + \hat{F}_{jl})\rangle +\\ &\langle\partial_k\partial_i\hat{A}_b, \hat{A}_j, \hat{A}_l\rangle + 2\langle\partial_k\hat{A}_i, \partial_b\hat{A}_j, \hat{A}_l\rangle] + \mathcal{O}(A^4)\end{aligned} \tag{5.47}$$

On solving these two equations consistently, we get $\hat{F}_{ab}$ in terms of $F_{ab}$ and $\hat{A}_b$ in terms of $A_b$, and they are:

$$\begin{aligned}&\hat{F}_{ab} = F_{ab} - \theta^{cd}[\langle A_c, \partial_d F_{ab}\rangle - \langle F_{ac}, F_{bd}\rangle]\\ &-\frac{1}{2}\theta^{cd}\theta^{ef}[\partial_c\partial_e\langle F_{ab}, A_d, A_f\rangle - \partial_e\langle F_{cd}, F_{ab}, A_f\rangle + 2\partial_e\langle F_{ac}, F_{bd}, A_f\rangle]\\ &+\theta^{cd}\theta^{ef}[\frac{1}{2}\langle F_{ac}, F_{bd}, F_{ef}\rangle - \frac{1}{8}\langle F_{ab}, F_{cd}, F_{ef}\rangle - \frac{1}{4}\langle F_{ab}, F_{de}, F_{cf}\rangle + \langle F_{ce}, F_{af}, F_{bd}\rangle]\\ &-\theta^{cd}\theta^{ef}[-\langle\langle A_e, \partial_f A_c\rangle, \partial_d F_{ab}\rangle - \frac{1}{2}\langle\langle\partial_d A_e, \partial_c A_f\rangle, F_{ab}\rangle + \frac{1}{2}\langle\langle A_e, \partial_c A_f\rangle, \partial_d F_{ab}\rangle\\ &-\langle A_c, \langle\partial_d A_e, \partial_f F_{ab}\rangle\rangle - \langle A_c, \langle A_e, \partial_d\partial_f F_{ab}\rangle\rangle + \langle A_c, \langle\partial_d F_{ae}, F_{bf}\rangle\rangle + \langle A_c, \langle F_{ae}, \partial_d F_{bf}\rangle\rangle\\ &+\langle F_{ac}, \langle A_e, \partial_f F_{bd}\rangle\rangle - \langle F_{ac}, \langle F_{be}, F_{df}\rangle\rangle + \langle\langle A_e, \partial_f F_{ac}\rangle, F_{bd}\rangle - \langle\langle F_{ae}, F_{cf}\rangle, F_{bd}\rangle] + \mathcal{O}(A^4)\end{aligned} \tag{5.48}$$

and

$$\hat{A}_b = A_b - \theta^{ij}\langle A_i, \partial_j A_b\rangle + \frac{1}{2}\theta^{ij}\langle A_i, \partial_b A_j\rangle$$
$$-\frac{1}{2}\theta^{ij}\theta^{kl}[-2\langle A_i, \partial_k A_b, \partial_j A_l\rangle + \langle A_i, \partial_k A_b, \partial_l A_j\rangle + \langle \partial_k\partial_i A_b, A_j, A_l\rangle + 2\langle \partial_k A_i, \partial_b A_j, A_l\rangle]$$
$$+\theta^{ij}\theta^{kl}[\langle A_i, \partial_j\langle A_k, \partial_l A_b\rangle\rangle + \langle\langle A_k, \partial_l A_i\rangle, \partial_j A_b\rangle - \frac{1}{2}\langle A_i, \partial_j\langle A_k, \partial_b A_l\rangle\rangle - \frac{1}{2}\langle\langle A_k, \partial_i A_l\rangle, \partial_j A_b\rangle$$
$$-\frac{1}{2}\langle A_i, \partial_b\langle A_k, \partial_l A_j\rangle\rangle + \frac{1}{4}\langle A_i, \partial_b\langle A_k, \partial_j A_l\rangle\rangle - \frac{1}{2}\langle\langle A_k, \partial_l A_i\rangle, \partial_b A_j\rangle$$
$$+\frac{1}{4}\langle\langle A_k, \partial_i A_l\rangle, \partial_b A_j\rangle - \frac{1}{2}\langle A_i, \langle \partial_k A_j, \partial_l A_b\rangle\rangle] + \mathcal{O}(A^4) \tag{5.49}$$

# 6 Appendix B

In this appendix, we shall write down explicitly the form of $*_2$, $*_3$, and $*_4$ etc, and also their infinitesimal form. The form of $*_2$ is :

$$\langle f, g\rangle = \frac{\sin(\frac{\partial_1\wedge\partial_2}{2})}{\frac{\partial_1\wedge\partial_2}{2}} f_1 g_2|_{1=2} \tag{6.50}$$

Where $\partial_1 \wedge \partial_2 = \partial_{1i}\theta^{ij}\partial_{2j}$, and $f_1 = f(x_1)$. It's infinitesimal form, up to 8-derivative is:

$$\langle f, g\rangle = fg - \frac{\theta^{ij}\theta^{kl}}{24}\partial_i\partial_k f\partial_j\partial_l g + \frac{\theta^{ij}\theta^{kl}\theta^{mn}\theta^{pq}}{1920}\partial_i\partial_k\partial_m\partial_p f\partial_j\partial_l\partial_n\partial_q g - \dots \tag{6.51}$$

The form of $*_3$ is:

$$\langle f, g, h\rangle = \{\frac{\sin(\frac{\partial_2\wedge\partial_3}{2})\sin(\frac{\partial_1\wedge(\partial_2+\partial_3)}{2})}{\frac{(\partial_1+\partial_2)\wedge\partial_3}{2}\frac{\partial_1\wedge(\partial_2+\partial_3)}{2}} + \frac{\sin(\frac{\partial_1\wedge\partial_3}{2})\sin(\frac{\partial_2\wedge(\partial_1+\partial_3)}{2})}{\frac{(\partial_1+\partial_2)\wedge\partial_3}{2}\frac{\partial_2\wedge(\partial_1+\partial_3)}{2}}\} f_1 g_2 h_3 \tag{6.52}$$

The infinitesimal form of this, up to 8-derivative is:

$$fgh - \frac{\theta^{ij}\theta^{kl}}{24}\{\partial_i\partial_k f\partial_j\partial_l gh + \partial_i\partial_k fg\partial_j\partial_l h + f\partial_i\partial_k g\partial_j\partial_l h\}$$
$$+\theta^{ij}\theta^{kl}\theta^{mn}\theta^{pq}\{\frac{1}{1920}(\partial_i\partial_k\partial_m\partial_p f\partial_j\partial_l\partial_n\partial_q gh + \partial_i\partial_k\partial_m\partial_p fg\partial_j\partial_l\partial_n\partial_q h$$
$$+f\partial_i\partial_k\partial_m\partial_p g\partial_j\partial_l\partial_n\partial_q h) + \frac{1}{576}(\partial_i\partial_k\partial_m\partial_p f\partial_j\partial_l g\partial_n\partial_q h + \partial_i\partial_k f\partial_m\partial_p g\partial_j\partial_l\partial_n\partial_q h$$
$$+\partial_i\partial_k f\partial_j\partial_l\partial_n\partial_q g\partial_m\partial_p h) + \frac{1}{720}(\partial_i\partial_k\partial_p f\partial_j\partial_m\partial_q g\partial_l\partial_n h - \partial_i\partial_k\partial_p f\partial_j\partial_m g\partial_n\partial_l\partial_q h$$
$$+\partial_i\partial_k f\partial_j\partial_m\partial_p g\partial_l\partial_n\partial_q h)\}\dots \tag{6.53}$$

The expression of the $*_4$ is:

$$\langle f,g,h,p\rangle = \frac{\sin(\frac{\partial_1\wedge\partial_4}{2})}{\frac{(\partial_1+\partial_2+\partial_3)\wedge\partial_4}{2}}(\frac{\sin(\frac{(\partial_1+\partial_4)\wedge\partial_3}{2})\sin(\frac{(\partial_1+\partial_3+\partial_4)\wedge\partial_2}{2})}{\frac{(\partial_1+\partial_2+\partial_4)\wedge\partial_3}{2}\frac{(\partial_1+\partial_3+\partial_4)\wedge\partial_2}{2}}+$$
$$\frac{\sin(\frac{\partial_2\wedge\partial_3}{2})\sin(\frac{(\partial_1+\partial_4)\wedge(\partial_2+\partial_3)}{2})}{\frac{(\partial_1+\partial_2+\partial_4)\wedge\partial_3}{2}\frac{(\partial_1+\partial_4)\wedge(\partial_2+\partial_3)}{2}})+\frac{\sin(\frac{\partial_2\wedge\partial_4}{2})}{\frac{(\partial_1+\partial_2+\partial_3)\wedge\partial_4}{2}}$$
$$(\frac{\sin(\frac{\partial_1\wedge\partial_3}{2})\sin(\frac{(\partial_1+\partial_3)\wedge(\partial_2+\partial_4)}{2})}{\frac{(\partial_1+\partial_2+\partial_4)\wedge\partial_3}{2}(\frac{(\partial_1+\partial_3)\wedge(\partial_2+\partial_4)}{2})}+\frac{\sin(\frac{(\partial_2+\partial_4)\wedge\partial_3}{2})\sin(\frac{\partial_1\wedge(\partial_2+\partial_3+\partial_4)}{2})}{\frac{(\partial_1+\partial_2+\partial_4)\wedge\partial_3}{2}\frac{\partial_1\wedge(\partial_2+\partial_3+\partial_4)}{2}})$$
$$+\frac{\sin(\frac{\partial_3\wedge\partial_4}{2})}{\frac{(\partial_1+\partial_2+\partial_3)\wedge\partial_4}{2}}(\frac{\sin(\frac{\partial_1\wedge(\partial_3+\partial_4)}{2})\sin(\frac{(\partial_1+\partial_3+\partial_4)\wedge\partial_2}{2})}{\frac{(\partial_1+\partial_2)\wedge(\partial_3+\partial_4)}{2}\frac{(\partial_1+\partial_3+\partial_4)\wedge\partial_2}{2}}$$
$$+\frac{\sin(\frac{\partial_2\wedge(\partial_3+\partial_4)}{2})\sin(\frac{\partial_1+(\partial_2+\partial_3+\partial_4)}{2})}{\frac{(\partial_1+\partial_2)\wedge(\partial_3+\partial_4)}{2}\frac{\partial_1\wedge(\partial_2+\partial_3+\partial_4)}{2}})f_1g_2h_3p_4|_{1=2=3=4} \tag{6.54}$$

The infinitesimal form of this, up to 8-derivative is:

$$\begin{aligned}
&fghp-\frac{\theta^{ij}\theta^{kl}}{24}\{\partial_i\partial_kf\partial_j\partial_lghp+\partial_i\partial_kfg\partial_j\partial_lhp+\partial_i\partial_kfgh\partial_j\partial_lp+f\partial_i\partial_kg\partial_j\partial_lhp\\
&+f\partial_i\partial_kgh\partial_j\partial_lp+fg\partial_i\partial_kh\partial_j\partial_lp\}+\frac{\theta^{ij}\theta^{kl}\theta^{mn}\theta^{rs}}{1920}\{\partial_i\partial_k\partial_m\partial_rf\partial_j\partial_l\partial_n\partial_sghp\\
&+\partial_i\partial_k\partial_m\partial_rfg\partial_j\partial_l\partial_n\partial_shp+\partial_i\partial_k\partial_m\partial_rfgh\partial_j\partial_l\partial_n\partial_sp+f\partial_i\partial_k\partial_m\partial_rg\partial_j\partial_l\partial_n\partial_shp\\
&+f\partial_i\partial_k\partial_m\partial_rgh\partial_j\partial_l\partial_n\partial_sp+fg\partial_i\partial_k\partial_m\partial_rh\partial_j\partial_l\partial_n\partial_sp\}+\frac{\theta^{ij}\theta^{kl}\theta^{mn}\theta^{rs}}{576}\\
&\{\partial_i\partial_k\partial_m\partial_rf\partial_j\partial_lg\partial_n\partial_sp+\partial_i\partial_k\partial_m\partial_rf\partial_j\partial_lgh\partial_n\partial_sp+\partial_i\partial_kf\partial_j\partial_l\partial_n\partial_sg\partial_m\partial_rhp\\
&+\partial_i\partial_kf\partial_j\partial_l\partial_n\partial_sgh\partial_m\partial_rp+\partial_i\partial_kf\partial_j\partial_lg\partial_m\partial_rh\partial_n\partial_sp+\partial_i\partial_k\partial_m\partial_rfg\partial_j\partial_lh\partial_n\partial_sp\\
&+\partial_i\partial_kf\partial_m\partial_rg\partial_j\partial_l\partial_n\partial_shp+\partial_i\partial_kf\partial_m\partial_rg\partial_j\partial_lh\partial_n\partial_sp+\partial_i\partial_kfg\partial_j\partial_l\partial_n\partial_sh\partial_m\partial_rp\\
&+\partial_i\partial_kf\partial_m\partial_rg\partial_n\partial_sh\partial_j\partial_lp+\partial_i\partial_kf\partial_m\partial_rgh\partial_j\partial_l\partial_n\partial_sp+\partial_i\partial_kfg\partial_m\partial_rh\partial_j\partial_l\partial_n\partial_sp\\
&+f\partial_i\partial_k\partial_m\partial_rg\partial_j\partial_lh\partial_n\partial_sp+f\partial_i\partial_kg\partial_j\partial_l\partial_n\partial_sh\partial_m\partial_rp+f\partial_i\partial_kg\partial_m\partial_rh\partial_j\partial_l\partial_n\partial_sp\}\\
&+\frac{\theta^{ij}\theta^{kl}\theta^{mn}\theta^{rs}}{720}\{\partial_i\partial_k\partial_mf\partial_j\partial_l\partial_rg\partial_n\partial_shp+\partial_i\partial_k\partial_mf\partial_j\partial_l\partial_rgh\partial_n\partial_sp\\
&+\partial_i\partial_k\partial_mfg\partial_j\partial_l\partial_rh\partial_n\partial_sp-\partial_i\partial_k\partial_mf\partial_n\partial_rg\partial_j\partial_l\partial_shp-\partial_i\partial_k\partial_mf\partial_n\partial_rgh\partial_j\partial_l\partial_sp\\
&-\partial_i\partial_k\partial_mfg\partial_n\partial_rh\partial_j\partial_l\partial_sp+f\partial_i\partial_k\partial_mg\partial_j\partial_l\partial_rh\partial_n\partial_sp\partial_m\partial_rf\partial_i\partial_k\partial_ng\partial_j\partial_l\partial_shp\\
&+\partial_m\partial_rf\partial_i\partial_k\partial_ngh\partial_j\partial_l\partial_sp-f\partial_i\partial_k\partial_mg\partial_n\partial_rh\partial_j\partial_l\partial_sp+f\partial_m\partial_rg\partial_i\partial_k\partial_nh\partial_j\partial_l\partial_sp\\
&+\partial_m\partial_rfg\partial_i\partial_k\partial_nh\partial_j\partial_l\partial_sp\}+\frac{\theta^{ij}\theta^{kl}\theta^{mn}\theta^{rs}}{720}\{\partial_i\partial_kf\partial_m\partial_rg\partial_j\partial_nh\partial_l\partial_sp\\
&-\partial_i\partial_kf\partial_j\partial_mg\partial_n\partial_rh\partial_l\partial_sp+\partial_i\partial_kf\partial_j\partial_mg\partial_l\partial_rh\partial_n\partial_sp\}
\end{aligned}\tag{6.55}$$

From now onwards, we shall write down only the infinitesimal form of $*_n$ within $*_m$ for $n\leq m$. The infinitesimal form of $\langle\langle f,g\rangle,h\rangle$,up to 4-derivative is:

$$\langle\langle f,g\rangle,h\rangle \;=\; fgh-\frac{1}{24}\theta^{ij}\theta^{kl}[(\partial_i\partial_kf)(\partial_j\partial_lg)h+(\partial_i\partial_kf)g(\partial_j\partial_lh)+f(\partial_i\partial_kg)(\partial_j\partial_lh)$$

$$+2(\partial_i f)(\partial_k g)(\partial_j \partial_l h)] + \dots \tag{6.56}$$

The infinitesimal form of $\langle\langle f, g, h\rangle, Q\rangle$, up to 4-derivative is:

$$\begin{aligned} &fghQ - \frac{\theta^{ij}\theta^{kl}}{24}\{\partial_i\partial_k f\partial_j\partial_l ghQ + \partial_i\partial_k fg\partial_j\partial_l hQ + \partial_i\partial_k fgh\partial_j\partial_l Q \\ &+f\partial_i\partial_k g\partial_j\partial_l hQ + f\partial_i\partial_k gh\partial_j\partial_l Q + fg\partial_i\partial_k h\partial_j\partial_l Q \\ &+2\partial_i f\partial_k gh\partial_j\partial_l Q + 2\partial_i fg\partial_k h\partial_j\partial_l Q + 2f\partial_i g\partial_k h\partial_j\partial_l Q\} \end{aligned} \tag{6.57}$$

The infinitesimal form of $\langle f, \langle\langle g, h\rangle, Q\rangle\rangle$, up to 4-derivative is:

$$\begin{aligned} &fghQ - \frac{\theta^{ij}\theta^{kl}}{24}\{\partial_i\partial_k f\partial_j\partial_l ghQ + \partial_i\partial_k fg\partial_j\partial_l hQ + \partial_i\partial_k fgh\partial_j\partial_l Q \\ &+f\partial_i\partial_k g\partial_j\partial_l hQ + f\partial_i\partial_k gh\partial_j\partial_l Q + fg\partial_i\partial_k h\partial_j\partial_l Q + 2\partial_i\partial_k f\partial_j g\partial_l hQ \\ &+2\partial_i\partial_k f\partial_j gh\partial_l Q + 2\partial_i\partial_k fg\partial_j h\partial_l Q + 2f\partial_i g\partial_k h\partial_j\partial_l Q\} \end{aligned} \tag{6.58}$$

The infinitesimal form of $\langle\langle f, g\rangle, \langle h, Q\rangle\rangle$, up to 4-derivative is:

$$\begin{aligned} &fghQ - \frac{\theta^{ij}\theta^{kl}}{24}\{\partial_i\partial_k f\partial_j\partial_l ghQ + \partial_i\partial_k fg\partial_j\partial_l hQ + \partial_i\partial_k fgh\partial_j\partial_l Q \\ &+f\partial_i\partial_k g\partial_j\partial_l hQ + f\partial_i\partial_k gh\partial_j\partial_l Q + fg\partial_i\partial_k h\partial_j\partial_l Q + 2\partial_i\partial_k fg\partial_j h\partial_l Q \\ &+2\partial_i f\partial_k g\partial_j\partial_l hQ + 2\partial_i f\partial_k g\partial_j h\partial_l Q + 2\partial_i f\partial_k g\partial_l h\partial_j Q + 2\partial_i f\partial_k gh\partial_j\partial_l Q \\ &+2f\partial_i\partial_k g\partial_j h\partial_l Q\} \end{aligned} \tag{6.59}$$

The infinitesimal form of $\langle\langle f, g\rangle, h, Q\rangle$, up to 4-derivative is:

$$\begin{aligned} &fghQ - \frac{\theta^{ij}\theta^{kl}}{24}\{\partial_i\partial_k f\partial_j\partial_l ghQ + \partial_i\partial_k fg\partial_j\partial_l hQ + \partial_i\partial_k fgh\partial_j\partial_l Q \\ &+f\partial_i\partial_k g\partial_j\partial_l hQ + f\partial_i\partial_k gh\partial_j\partial_l Q + fg\partial_i\partial_k h\partial_j\partial_l Q + 2\partial_i f\partial_k g\partial_j\partial_l hQ \\ &+2\partial_i f\partial_k gh\partial_j\partial_l Q\} \end{aligned} \tag{6.60}$$

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